A HERBRAND-RIBET THEOREM FOR FUNCTION FIELDS

LENNY TAELMAN

Abstract. We prove a function field analogue of the Herbrand-Ribet theorem on cyclotomic number fields. The Herbrand-Ribet theorem can be interpreted as a result about cohomology with μ_p-coefficients over the splitting field of μ_p, and in our analogue both occurrences of μ_p are replaced with the $\mathfrak{p}$-torsion scheme of the Carlitz module for a prime $\mathfrak{p}$ in $\mathbf{F}_q[t]$.

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1. Introduction and statement of the theorem

Let p be a prime number, $F = \mathbf{Q}(\zeta_p)$ the p-th cyclotomic number field and $\operatorname{Pic}\mathcal{O}_F$ its class group. Then $\mathbf{F}_p \otimes_{\mathbf{Z}} \operatorname{Pic}\mathcal{O}_F$ decomposes in eigenspaces under the action of the Galois group $\operatorname{Gal}(F/\mathbf{Q})$ as

$$\mathbf{F}_p \otimes_{\mathbf{Z}} \operatorname{Pic}\mathcal{O}_F = \bigoplus_{n=1}^{p-1} \left(\mathbf{F}_p \otimes_{\mathbf{Z}} \operatorname{Pic}\mathcal{O}_F\right)(\omega^n)$$

where $\omega\colon \operatorname{Gal}(F/\mathbf{Q}) \to \mathbf{F}_p^\times$ is the cyclotomic character.

If n is a nonnegative integer we denote by B_n the n-th Bernoulli number, defined by the identity

$$\frac{z}{\exp z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}.$$

If n is smaller than p then B_n is p-integral. The *Herbrand-Ribet theorem* [9] [14] states that if n is even and $1 < n < p$ then

$$\left(\mathbf{F}_p \otimes_{\mathbf{Z}} \operatorname{Pic}\mathcal{O}_F\right)(\omega^{1-n}) \neq 0 \text{ if and only if } p \mid B_n.$$

The *Kummer-Vandiver conjecture* asserts that for all odd n we have

$$(\mathbf{F}_p \otimes_{\mathbf{Z}} \operatorname{Pic} \mathcal{O}_F)\,(\omega^{1-n}) = 0.$$

In this paper we will state and prove a function field analogue of the Herbrand-Ribet theorem and state an analogue of the Kummer-Vandiver conjecture.

Let k be a finite field of q elements and $A = k[t]$ the polynomial ring in one variable t over k. Let K be the fraction field of A.

Definition 1. The *Carlitz module* is the A-module scheme C over $\operatorname{Spec} A$ whose underlying k-vectorspace scheme is the additive group $\mathbf{G}_a$ and whose $k[t]$-module structure is given by the k-algebra homomorphism

$$\varphi\colon A \to \operatorname{End}(\mathbf{G}_a),\ t \mapsto t + F,$$

where F is the q-th power Frobenius endomorphism of $\mathbf{G}_a$.

The Carlitz module is in many ways an A-module analogue of the $\mathbf{Z}$-module scheme $\mathbf{G}_m$. For example, the $\operatorname{Gal}(K^{\mathrm{sep}}/K)$-action on torsion points is formally similar to the $\operatorname{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$-action on roots of unity:

Proposition 1 ([7, §7.5])**.** *Let $\mathfrak{p} \subset A$ be a nonzero prime ideal, then $C[\mathfrak{p}](K^{\mathrm{sep}}) \cong A/\mathfrak{p}$ and the resulting Galois representation*

$$\rho\colon \operatorname{Gal}(K^{\mathrm{sep}}/K) \longrightarrow (A/\mathfrak{p})^{\times}.$$

satisfies

1. *if a prime $\mathfrak{q} \subset A$ is coprime with $\mathfrak{p}$ then ρ is unramified at $\mathfrak{q}$ and maps a Frobenius element to the class in $(A/\mathfrak{p})^{\times}$ of the monic generator of $\mathfrak{q}$;*
2. *$\rho(D_\infty) = \rho(I_\infty) = k^{\times}$;*
3. *$\rho(D_{\mathfrak{p}}) = \rho(I_{\mathfrak{p}}) = (A/\mathfrak{p})^{\times}$,*

where the D's and I's denote decomposition and inertia subgroups. □

Now fix a nonzero prime ideal $\mathfrak{p} \subset A$ of degree d. Let L be the splitting field of ρ. Then L/K is unramified outside $\mathfrak{p}$ and ∞, and ρ induces an isomorphism $\chi\colon G = \operatorname{Gal}(L/K) \xrightarrow{\sim} (A/\mathfrak{p})^{\times}$.

Let R be the normalization of A in L and $Y = \operatorname{Spec} R$. Let Y_{fl} be the flat site on Y: the category of schemes locally of finite type over Y, with covering families being the jointly surjective families of flat morphisms.

The $\mathfrak{p}$-torsion $C[\mathfrak{p}]$ of C is a finite flat group scheme of rank q^d over $\operatorname{Spec} A$. Let $C[\mathfrak{p}]^{\mathrm{D}}$ be the Cartier dual of $C[\mathfrak{p}]$ and consider the decomposition

$$\mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}}) = \bigoplus_{n=1}^{q^d-1} \mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}})(\chi^n)$$

of the $A/\mathfrak{p}$-vector space $\mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}})$ under the natural action of G.

Our analogue of the Herbrand-Ribet theorem will give a criterion for the vanishing of some of these eigenspaces in terms of divisibility by $\mathfrak{p}$ of the so-called Bernoulli-Carlitz numbers, which we now define.

The *Carlitz exponential* is the unique power series $e(z) \in K[[z]]$ which satisfies

1. $e(z) = z + e_1 z^q + e_2 z^{q^2} + \cdots$ with $e_i \in K$;
2. $e(tz) = e(z)^q + te(z)$.

The Carlitz exponential converges on any finite extension of K_∞ and on an algebraic closure $\bar{K}_\infty$ it defines a surjective homomorphism of A-modules

$$e\colon \bar{K}_\infty \twoheadrightarrow C(\bar{K}_\infty)$$

whose kernel is discrete and free of rank 1. We define $BC_n \in K$ by the power series identity

$$\frac{z}{e(z)} = \sum_{n=0}^{\infty} BC_n z^n.$$

If n is not divisible by $q-1$ then BC_n is zero. If n is less than q^d then BC_n is $\mathfrak{p}$-integral.

Theorem 1. *Let $0 < n < q^d - 1$ be divisible by $q-1$. Then $\mathfrak{p}$ divides BC_n if and only if $\mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}})(\chi^{n-1})$ is nonzero.*

This is the analogue of the Herbrand-Ribet theorem. The proof is given in section 4, modulo auxiliary results which are proven in sections 6–9.

In this context a natural analogue of the Kummer-Vandiver conjecture is the following:

Question 1. *Does $\mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}})(\chi^{n-1})$ vanish if n is not divisible by $q-1$?*

By computer calculation we have verified that these groups indeed vanish for small q and primes $\mathfrak{p}$ of small degree, *see* §2. However, if one believes in a function field version of Washington's heuristics [18, §9.3] then one should expect that counterexamples do exist, but are very sparse, making it difficult to obtain convincing numerical evidence towards Question 1.

Remark 1. Our BC_n differ from the commonly used *Bernoulli-Carlitz* numbers by a *Carlitz factorial* factor (see for example [7, §9.2]). This factor is innocent for our purposes since it is a unit at $\mathfrak{p}$ for $n < q^d$.

Remark 2. Let p be an odd prime number, $F = \mathbf{Q}(\zeta_p)$ and $D = \operatorname{Spec} \mathcal{O}_F$. Global duality [10] provides a perfect pairing between

$$\mathbf{F}_p \otimes_{\mathbf{Z}} \operatorname{Pic} D = \operatorname{Ext}^2_{D_{\mathrm{et}}}(\mathbf{Z}/p\mathbf{Z}, \mathbf{G}_{m,D})$$

and

$$\mathrm{H}^1(D_{\mathrm{et}}, \mathbf{Z}/p\mathbf{Z}) = \mathrm{H}^1(D_{\mathrm{fl}}, \mathbf{Z}/p\mathbf{Z}).$$

The Herbrand-Ribet theorem thus says that (for $1 < n < p-1$ even)

$$p \mid B_n \text{ if and only if } \mathrm{H}^1(D_{\mathrm{fl}}, \mu_p^{\mathrm{D}})(\chi^{n-1}) \neq 0,$$

in perfect analogy with the statement of Theorem 1.

Remark 3. The analogy goes even further. In [16] and [15] we have defined a finite A-module $H(C/R)$, analogue of the class group $\operatorname{Pic} \mathcal{O}_F$, and although we will not use this in the proof of Theorem 1, we show in Section 10 of this paper that there are canonical isomorphisms

$$A/\mathfrak{p} \otimes_A H(C/R) \xrightarrow{\sim} \operatorname{Hom}(\mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}}), \mathbf{F}_p).$$

Remark 4. A more naive attempt to obtain a function field analogue of the Herbrand-Ribet theorem would be to compare the $\mathfrak{p}$-divisibility of the Bernoulli-Carlitz numbers with the p-torsion of the divisor class groups of Y and L (where p is the characteristic of k). In other words, to consider cohomology with μ_p-coefficients

on the curves defined by the splitting of $C[\mathfrak{p}]$. Several results of this kind have in fact been obtained by Goss [6], Gekeler [5], Okada [12], and Anglès [2], but there appears to be no complete analogue of the Herbrand-Ribet theoreom in this context.

In the proof of Theorem 1 we will see that the A-module $\mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}})$ and the group $(\operatorname{Pic} Y)[p]$ are related, and this relationship might shed some new light on these older results.

Remark 5. I do not know if there is a relation between Question 1 and Anderson's analogue of the Kummer-Vandiver conjecture [1].

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2. Tables of small irregular primes

The results of section 10 indicate a method for computing the modules $\mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}})$ with their G-action in terms of finite-dimensional vector spaces of differential forms on the compactification X of Y.

Assisted by the computer algebra package MAGMA we were able to compute them in the following ranges:

(1) $q = 2$ and $\deg \mathfrak{p} \leq 5$;
(2) $q = 3$ and $\deg \mathfrak{p} \leq 4$;
(3) $q = 4$ and $\deg \mathfrak{p} \leq 3$;
(4) $q = 5$ and $\deg \mathfrak{p} \leq 3$.

In all these cases $\mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}})$ turns out to be at most one-dimensional, and to fall in the χ^{n-1}-component with n divisible by $q-1$ (and hence with $\mathfrak{p}$ dividing BC_n.) In particular we have not found any counterexamples to Question 1.

In tables 1–3 we list all cases where the cohomology group is nontrivial. For $q = 5$ and $\deg \mathfrak{p} \leq 3$ the group turns out to vanish. In the middle columns, only n in the range $1 \leq n < q^{\deg \mathfrak{p}}$ are printed.

$\mathfrak{p}$	$\{n : \mathfrak{p} \mid BC_n\}$	$\dim \mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}})$
$(t^4 + t + 1)$	$\{9\}$	1

Table 1. All irregular primes in $\mathbf{F}_2[t]$ of degree at most 5

3. Notation and conventions

Basic setup. k is a finite field of q elements, p its characteristic. $A = k[t]$ and $\mathfrak{p} \subset A$ a nonzero prime. These data are fixed throughout the text. We denote by d the degree of $\mathfrak{p}$, so that $A/\mathfrak{p}$ is a field of q^d elements.

The Carlitz module. The Carlitz module is the A-module scheme C over $\operatorname{Spec} A$ defined in Definition 1.

Cyclotomic curves and fields. K is the fraction field of A, and L/K the splitting field of $C[\mathfrak{p}]_K$. The integral closure of A in L is denoted by R, and $Y = \operatorname{Spec} R$. We denote by $\mathfrak{P} \subset R$ the unique prime lying above $\mathfrak{p} \subset A$.

Sites. For any scheme S we denote by S_{et} the *small étale site* on S and by S_{fl} the *flat site* in the sense of [11]: the category of schemes locally of finite type over S

$\mathfrak{p}$	$\{n : \mathfrak{p} \mid BC_n\}$	$\dim \mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}})$
$(t^3 - t + 1)$	$\{10\}$	1
$(t^3 - t - 1)$	$\{10\}$	1
$(t^4 - t^3 + t^2 + 1)$	$\{40\}$	1
$(t^4 - t^2 - 1)$	$\{32\}$	1
$(t^4 - t^3 - t^2 + t - 1)$	$\{32\}$	1
$(t^4 + t^3 + t^2 + 1)$	$\{40\}$	1
$(t^4 + t^3 - t^2 - t - 1)$	$\{32\}$	1
$(t^4 + t^2 - 1)$	$\{40\}$	1

TABLE 2. All irregular primes in $\mathbf{F}_3[t]$ of degree at most 4

$\mathfrak{p}$	$\{n : \mathfrak{p} \mid BC_n\}$	$\dim \mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}})$
$(t^3 + t^2 + t + \alpha)$	$\{33\}$	1
$(t^3 + t^2 + t + \alpha^2)$	$\{33\}$	1
$(t^3 + \alpha)$	$\{33\}$	1
$(t^3 + \alpha^2)$	$\{33\}$	1
$(t^3 + \alpha^2 t^2 + \alpha t + \alpha^2)$	$\{33\}$	1
$(t^3 + \alpha t^2 + \alpha^2 t + \alpha)$	$\{33\}$	1
$(t^3 + \alpha t^2 + \alpha^2 t + \alpha^2)$	$\{33\}$	1
$(t^3 + \alpha^2 t^2 + \alpha t + \alpha)$	$\{33\}$	1

TABLE 3. All irregular primes in $\mathbf{F}_4[t]$ of degree at most 3 (with $\mathbf{F}_4 = \mathbf{F}_2(\alpha)$).

where covering families are jointly surjective families of flat morphisms. For every S there is a canonical morphism of sites $f\colon S_{\mathrm{fl}} \to S_{\mathrm{et}}$. Any commutative group scheme over S defines a sheaf of abelian groups on S_{fl} and on S_{et}.

Cartier dual. If G is a finite flat commutative group scheme, then G^{D} denotes the Cartier dual of G.

Frobenius and Cartier operators. For any k-scheme S we denote by

$$\mathrm{F}\colon \mathbf{G}_{a,S} \to \mathbf{G}_{a,S},\, x \mapsto x^q$$

the q-power Frobenius endomorphism of sheaves on S_{fl} or S_{et}, and by

$$\mathrm{c}\colon \Omega_S \to \Omega_S$$

the q-Cartier operator of sheaves on S_{et}. If $q = p^r$ with p prime this is the r-th power of the usual Cartier operator. The endomorphism c satisfies $\mathrm{c}(f^q\omega) = f\mathrm{c}(\omega)$ for all local sections f of $\mathcal{O}_S$ and ω of Ω_S. In particular it is k-linear.

4. Overview of the proof

Choose a generator λ of $C[\mathfrak{p}](L)$. It defines a map of finite flat group schemes

$$\lambda\colon (A/\mathfrak{p})_Y \longrightarrow C[\mathfrak{p}]_Y$$

which is an isomorphism over $Y - \mathfrak{P}$. It induces a map of Cartier duals

$$C[\mathfrak{p}]_Y^{\mathrm{D}} \longrightarrow (A/\mathfrak{p})_Y^{\mathrm{D}}$$

and a map on cohomology

$$\mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}}) \longrightarrow \mathrm{H}^1(Y_{\mathrm{fl}}, (A/\mathfrak{p})^{\mathrm{D}}).$$

This map is not G-equivariant (since λ is not G-invariant), but rather restricts for every n to a map

$$\mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}})(\chi^{n-1}) \xrightarrow{\lambda} \mathrm{H}^1(Y_{\mathrm{fl}}, (A/\mathfrak{p})^{\mathrm{D}})(\chi^n). \tag{1}$$

We will see in section 6 that there is a natural G-equivariant isomorphism

$$\mathrm{H}^1(Y_{\mathrm{fl}}, (A/\mathfrak{p})^{\mathrm{D}}) \xrightarrow{\sim} A/\mathfrak{p} \otimes_k \Omega_R^{\mathrm{c}=1}$$

where $\Omega_R^{\mathrm{c}=1}$ is the k-vector space of q-Cartier invariant Kähler differentials. Also, we will see that the Kummer sequence induces a short exact sequence

$$0 \longrightarrow A/\mathfrak{p} \otimes_{\mathbf{Z}} \Gamma(Y, \mathcal{O}_Y^\times) \xrightarrow{\mathrm{dlog}} A/\mathfrak{p} \otimes_k \Omega_R^{\mathrm{c}=1} \longrightarrow A/\mathfrak{p} \otimes_{\mathbf{F}_p} (\operatorname{Pic} Y)[p] \longrightarrow 0. \tag{2}$$

Note that the residue field of the completion $R_{\mathfrak{P}}$ is $A/\mathfrak{p}$, so $R_{\mathfrak{P}}$ is naturally an $A/\mathfrak{p}$-algebra. In particular, for all m the R-module $\Omega_R/\mathfrak{P}^m\Omega_R$ is naturally an $A/\mathfrak{p}$-module. Using this the quotient map $\Omega_R \twoheadrightarrow \Omega_R/\mathfrak{P}^m\Omega_R$ extends to an $A/\mathfrak{p}$-linear map

$$A/\mathfrak{p} \otimes_k \Omega_R \twoheadrightarrow \Omega_R/\mathfrak{P}^m\Omega_R.$$

In section §7 we will use the results on flat duality of Artin and Milne [3] to show the following.

Theorem 2. *For all n the sequence of $A/\mathfrak{p}$-vector spaces*

$$0 \longrightarrow \mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}})(\chi^{n-1}) \xrightarrow{\lambda} A/\mathfrak{p} \otimes_k \Omega_R^{\mathrm{c}=1}(\chi^n) \longrightarrow \Omega_R/\mathfrak{P}^{q^d}\Omega_R$$

is exact.

The function λ is invertible on $Y - \mathfrak{P}$. Consider the decomposition of $1 \otimes \lambda \in A/\mathfrak{p} \otimes_{\mathbf{Z}} \Gamma(Y - \mathfrak{P}, \mathcal{O}_Y^\times)$ in isotypical components:

$$1 \otimes \lambda = \sum_{n=1}^{q^d-1} \lambda_n \quad \text{with} \quad \lambda_n \in A/\mathfrak{p} \otimes_{\mathbf{Z}} \Gamma(Y - \mathfrak{P}, \mathcal{O}_Y^\times)(\chi^n).$$

The homomorphism $\mathrm{dlog}\colon R^\times \to \Omega_R$ extends to an $A/\mathfrak{p}$-linear map

$$A/\mathfrak{p} \otimes_{\mathbf{Z}} \Gamma(Y, \mathcal{O}_Y^\times) \longrightarrow \Omega_R.$$

Inspired by Okada's construction [12] of a Kummer homomorphism for function fields we prove in section 8 the following result.

Theorem 3. *If $1 \le n < q^d - 1$ then $\lambda_n \in A/\mathfrak{p} \otimes_{\mathbf{Z}} \Gamma(Y, \mathcal{O}_Y^\times)$ and the following are equivalent:*

1. $\mathfrak{p}$ *divides BC_n;*
2. $\mathrm{dlog}\,\lambda_n$ *lies in the kernel of $A/\mathfrak{p} \otimes_k \Omega_R \to \Omega_R/\mathfrak{P}^{q^d}\Omega_R$.*

It may (and does) happen that λ_n vanishes for some n divisible by $q-1$. However, the following theorem provides us with sufficient control over the vanishing of λ_n.

Theorem 4. *If n is divisible by $q-1$ but not by q^d-1 then the following are equivalent:*

1. $\lambda_n = 0$*;*
2. $A/\mathfrak{p} \otimes_{\mathbf{F}_p} (\operatorname{Pic} Y)[p](\chi^n) \neq 0.$

The proof is an adaptation of work of Galovich and Rosen [4], and uses L-functions in characteristic 0. It is given in section 9.

Assuming the three theorems above, we can now prove the main result.

Proof of Theorem 1. Assume $q-1$ divides n and $\mathfrak{p}$ divides BC_n. We need to show that $\mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}})(\chi^{1-n})$ is nonzero. Being a (component of) a differential logarithm $\operatorname{dlog}\lambda_n$ is Cartier-invariant and Theorem 3 tells us that

$$\operatorname{dlog}\lambda_n \in A/\mathfrak{p} \otimes_k \Omega_R^{\mathrm{c}=1}(\chi^n)$$

maps to 0 in $\Omega_R/\mathfrak{P}^{q^d}\Omega_R$. If $\lambda_n \neq 0$ then by Theorem 2 we conclude that $\mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}})(\chi^{n-1})$ is nonzero and we are done. So assume that $\lambda_n = 0$. Consider the short exact sequence (2). By Theorem 4 we have that

$$\dim_{A/\mathfrak{p}} A/\mathfrak{p} \otimes_{\mathbf{F}_p} (\operatorname{Pic} Y)[p](\chi^n) \geq 1,$$

and since $A/\mathfrak{p} \otimes_{\mathbf{Z}} \Gamma(Y, \mathcal{O}_Y^\times)(\chi^n)$ is one-dimensional, we find that

$$\dim_{A/\mathfrak{p}} A/\mathfrak{p} \otimes_k \Omega_R^{\mathrm{c}=1}(\chi^n) \geq 2.$$

But $\Omega_R/\mathfrak{P}^{q^d}\Omega_R(\chi^n)$ is one-dimensional, so it follows from Theorem 2 that

$$\mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}})(\chi^{n-1}) \neq 0.$$

Conversely, assume that $q-1$ divides n and $\mathfrak{p}$ does *not* divide BC_n. Then Theorem 3 guarantees that $\operatorname{dlog}\lambda_n$ is nonzero and it follows from Theorem 4 and the short exact sequence (2) that

$$\dim A/\mathfrak{p} \otimes_k \Omega_R^{\mathrm{c}=1}(\chi^n) = 1.$$

Therefore $A/\mathfrak{p} \otimes_k \Omega_R^{\mathrm{c}=1}(\chi^n)$ is generated by $\operatorname{dlog}\lambda_n$ and since the image of $\operatorname{dlog}\lambda_n$ in $\Omega_R/\mathfrak{P}^{q^d}\Omega_R$ is nonzero we conclude from Theorem 2 that $\mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]^{\mathrm{D}})(\chi^{n-1})$ vanishes. □

5. Flat duality

In this section we summarize some of the results of Artin and Milne [3] on duality for flat cohomology in characteristic p.

Let S be a scheme over k and $\mathcal{V}$ a quasi-coherent $\mathcal{O}_S$-module. Then the pull-back $F^*\mathcal{V}$ of $\mathcal{V}$ under $F\colon S \to S$ is a quasi-coherent $\mathcal{O}_S$-module and there is a k-linear (typically *not* $\mathcal{O}_S$-linear) isomorphism

$$F\colon \mathcal{V} \longrightarrow F^*\mathcal{V}$$

of sheaves on S_{fl}.

If S is smooth of relative dimension 1 over k then the q-Cartier operator induces a canonical map

$$\mathrm{c}\colon \mathcal{H}om(F^*\mathcal{V}, \Omega_{S/k}) \longrightarrow \mathcal{H}om(\mathcal{V}, \Omega_{S/k})$$

of sheaves on S_{et}.

Recall that the we denote the canonical map $S_{\mathrm{fl}} \to S_{\mathrm{et}}$ by f.

Theorem 5 (Artin & Milne)**.** *Let S be smooth of relative dimension 1 over $\operatorname{Spec} k$. Let*

$$0 \longrightarrow G \longrightarrow \mathcal{V} \xrightarrow{\alpha - F} F^*\mathcal{V} \longrightarrow 0 \tag{3}$$

be a short exact sequence of sheaves on S_{fl} with

(1) *$\mathcal{V}$ a locally free coherent $\mathcal{O}_S$-module;*

(2) $\alpha\colon \mathcal{V} \to F^*\mathcal{V}$ *a morphism of* $\mathcal{O}_S$*-modules.*

Then G *is a finite flat group scheme and there is a short exact sequence*

$$0 \longrightarrow \mathrm{R}^1 f_* G^{\mathrm{D}} \longrightarrow \mathcal{H}om(F^*\mathcal{V}, \Omega_{S/k}) \xrightarrow{\alpha - \mathrm{c}} \mathcal{H}om(\mathcal{V}, \Omega_{S/k}) \longrightarrow 0 \tag{4}$$

of sheaves on S_{et}*, functorial in (3). Moreover, for all* $i \neq 1$ *one has* $\mathrm{R}^i f_* G^{\mathrm{D}} = 0$*.*

Proof. Locally on S, we have that G is given as a closed subgroup scheme of $\mathbf{G}_a^n$ defined by equations of the form $FX - \alpha X = 0$. In particular G is flat of degree $q^{\mathrm{rk}\,\mathcal{V}}$. The Cartier dual G^{D} of G is a finite flat group scheme of height 1.

If q is prime then the existence of (4) is shown in [3, §2]. One can deduce the general case from this as follows. Assume n is a positive integer, and assume given a short exact sequence

$$0 \longrightarrow G \longrightarrow \mathcal{V} \xrightarrow{\alpha - F^n} (F^n)^*\mathcal{V} \longrightarrow 0$$

of sheaves on S_{fl}, with $\alpha\colon \mathcal{V} \to (F^n)^*\mathcal{V}$ a $\mathcal{O}_S$-linear map. Define

$$\mathcal{V}' := \mathcal{V} \oplus F^*\mathcal{V} \oplus \cdots \oplus (F^{n-1})^*\mathcal{V}.$$

The map α induces an $\mathcal{O}_S$-linear map

$$\alpha'\colon \mathcal{V}' \longrightarrow F^*\mathcal{V}'$$

defined by mapping the component $\mathcal{V}$ to the component $(F^n)^*\mathcal{V}$ using α, and mapping all other components to zero. We thus have a short exact sequence

$$0 \longrightarrow G \longrightarrow \mathcal{V}' \xrightarrow{\alpha' - F} F^*\mathcal{V}' \longrightarrow 0$$

and one deduces the theorem for F^n from the theorem for F. □

Example 5.1. If $k = \mathbf{F}_p$ then the Artin-Schreier exact sequence

$$0 \longrightarrow \mathbf{Z}/p\mathbf{Z} \longrightarrow \mathbf{G}_a \xrightarrow{1-F} \mathbf{G}_a \longrightarrow 0$$

on S_{fl} induces a dual exact sequence

$$0 \longrightarrow \mathrm{R}^1 f_* \mu_p \longrightarrow \Omega_{S/k} \xrightarrow{1-\mathrm{c}} \Omega_{S/k} \longrightarrow 0$$

on S_{et}, and the exact sequence

$$0 \longrightarrow \alpha_p \longrightarrow \mathbf{G}_a \xrightarrow{-F} \mathbf{G}_a \longrightarrow 0$$

on S_{fl} induces a dual exact sequence

$$0 \longrightarrow \mathrm{R}^1 f_* \alpha_p \longrightarrow \Omega_{S/k} \xrightarrow{-\mathrm{c}} \Omega_{S/k} \longrightarrow 0$$

on S_{et}.

6. Flat cohomology with $(A/\mathfrak{p})^{\mathrm{D}}$ coefficients

The constant sheaf $A/\mathfrak{p}$ on Y_{fl} has a resolution

$$0 \longrightarrow A/\mathfrak{p} \longrightarrow A/\mathfrak{p} \otimes_k \mathbf{G}_{a,Y} \xrightarrow{1-1\otimes F} A/\mathfrak{p} \otimes_k \mathbf{G}_{a,Y} \longrightarrow 0$$

so by Theorem 5 we have $\mathrm{R}^i f_*(A/\mathfrak{p})^{\mathrm{D}} = 0$ for $i \neq 1$, and $\mathrm{R}^1 f_*(A/\mathfrak{p})^{\mathrm{D}}$ sits in a short exact sequence

$$1 \longrightarrow \mathrm{R}^1 f_*(A/\mathfrak{p})^{\mathrm{D}} \longrightarrow A/\mathfrak{p} \otimes_k \Omega_Y \xrightarrow{1\otimes \mathrm{c} - 1} A/\mathfrak{p} \otimes_k \Omega_Y \longrightarrow 0$$

of sheaves on Y_{et}. Taking global sections now yields an isomorphism

$$\mathrm{H}^1(Y_{\mathrm{fl}}, (A/\mathfrak{p})^{\mathrm{D}}) \xrightarrow{\sim} A/\mathfrak{p} \otimes_k \Omega_{R/k}^{\mathrm{c}=1},$$

where $\Omega^{\mathrm{c}=1}_{R/k}$ denotes the k-vector space of Cartier-invariant Kähler differentials.

On the other hand, we have a natural isomorphism

$$(A/\mathfrak{p})^{\mathrm{D}} \xrightarrow{\sim} A/\mathfrak{p} \otimes_{\mathbf{F}_p} \mu_p,$$

of sheaves on Y_{fl} and the Kummer sequence

$$1 \longrightarrow \mu_p \longrightarrow \mathbf{G}_m \xrightarrow{p} \mathbf{G}_m \longrightarrow 1$$

gives rise to a short exact sequence

$$0 \longrightarrow A/\mathfrak{p} \otimes_{\mathbf{Z}} \Gamma(Y, \mathcal{O}_Y^{\times}) \longrightarrow \mathrm{H}^1(Y_{\mathrm{fl}}, (A/\mathfrak{p})^{\mathrm{D}}) \longrightarrow A/\mathfrak{p} \otimes_{\mathbf{F}_p} (\operatorname{Pic} Y)[p] \longrightarrow 0. \tag{5}$$

The proof of Theorem 5 shows that the resulting composed morphism

$$A/\mathfrak{p} \otimes_{\mathbf{Z}} \Gamma(Y, \mathcal{O}_Y^{\times}) \longrightarrow \mathrm{H}^1(Y_{\mathrm{fl}}, (A/\mathfrak{p})^{\mathrm{D}}) \xrightarrow{\sim} A/\mathfrak{p} \otimes_k \Omega_R^{\mathrm{c}=1}$$

is the map induced from

$$\mathrm{dlog}\colon \Gamma(Y, \mathcal{O}_Y^{\times}) \to \Omega_R^{\mathrm{c}=1} \colon u \mapsto \frac{\mathrm{d}u}{u},$$

so that (5) becomes the short exact sequence (2).

7. Comparing $(A/\mathfrak{p})^{\mathrm{D}}$ and $C[\mathfrak{p}]^{\mathrm{D}}$-coefficients

Choose a nonzero torsion point $\lambda \in C[\mathfrak{p}](L)$. Then λ defines a morphism $(A/\mathfrak{p})_Y \to C[\mathfrak{p}]_Y$ and hence a morphism of Cartier duals

$$C[\mathfrak{p}]_Y^{\mathrm{D}} \xrightarrow{\lambda} (A/\mathfrak{p})_Y^{\mathrm{D}}.$$

Let $\mathfrak{P} \in Y$ be the unique prime above $\mathfrak{p} \subset A$. We have $\mathfrak{P} = R\lambda$.

Proposition 2. *The sequence*

$$0 \longrightarrow \mathrm{R}^1 f_* C[\mathfrak{p}]^{\mathrm{D}} \xrightarrow{\lambda} \mathrm{R}^1 f_* (A/\mathfrak{p})^{\mathrm{D}} \longrightarrow \Omega_Y/\mathfrak{P}^{q^d}\Omega_Y \xrightarrow{1-\mathrm{c}^d} \Omega_Y/\mathfrak{P}\Omega_Y \longrightarrow 0, \tag{6}$$

of sheaves on Y_{et} is exact and if $i \neq 1$ then $\mathrm{R}^i f_* C[\mathfrak{p}]^{\mathrm{D}} = 0$.

Note that for all N the sheaf $\Omega_Y/\mathfrak{P}^N\Omega_Y$ on Y_{et} is naturally a sheaf of $A/\mathfrak{p}$-modules. The middle map in the proposition is the composition

$$\mathrm{R}^1 f_* (A/\mathfrak{p})^{\mathrm{D}} \longrightarrow A/\mathfrak{p} \otimes_k \Omega_Y \twoheadrightarrow \Omega_Y/\mathfrak{P}^{q^d}\Omega_Y.$$

Taking global sections in (6) we obtain an exact sequence of $A/\mathfrak{p}$-vector spaces

$$0 \longrightarrow \mathrm{H}^1(Y_{\mathrm{fl}}, C[\mathfrak{p}]_Y^{\mathrm{D}}) \xrightarrow{\lambda} A/\mathfrak{p} \otimes_k \Omega_R^{\mathrm{c}=1} \longrightarrow \Omega_R/\mathfrak{P}^{q^d}\Omega_R$$

and considering the G-action on λ we see that Proposition 2 implies Theorem 2.

As one may expect, the proof of Proposition 2 relies on a careful analysis of the group scheme $C[\mathfrak{p}]_Y$ near the prime $\mathfrak{P}$.

Let $\bar{s} \to Y$ be a geometric point lying above $\mathfrak{P} \in Y$,

Lemma 1. *There is an étale neighborhood $V \to Y$ of $\bar{s}$ and a short exact sequence*

$$0 \longrightarrow C[\mathfrak{p}]_V \longrightarrow \mathbf{G}_{a,V} \xrightarrow{\lambda^{q^d-1} - F^d} \mathbf{G}_{a,V} \longrightarrow 0$$

of sheaves of $A/\mathfrak{p}$-vector spaces on V_{fl}.

Proof. Let $\mathcal{O}_{Y,\bar{s}}$ be the étale stalk of $\mathcal{O}_Y$ at $\bar{s}$ (a strict henselization of $\mathcal{O}_{Y,\mathfrak{P}}$) and let $S = \operatorname{Spec} \mathcal{O}_{Y,\bar{s}}$. We have that $C[\mathfrak{p}]_S$ is a finite flat $A/\mathfrak{p}$-vector space scheme of rank q^d over S, étale over the generic fibre. Such vector space schemes have been classified by Raynaud [13, §1.5] (generalizing the results of Oort and Tate [17]). Let $q = p^r$ with $p = \operatorname{char} k$, then the classification says that $C[\mathfrak{p}]_S$ is a subgroupscheme of $\mathbf{G}_a^{rd}$ given by equations

$$X_i^p = a_i X_{i+1}$$

for some $a_i \in \mathcal{O}_{Y,\bar{s}}$, and where the index i runs over $\mathbf{Z}/rd\mathbf{Z}$. Since the special fibre of $C[\mathfrak{p}]_S$ is the kernel of F^d on $\mathbf{G}_a$, we find that all but one a_i are units. In particular, we can eliminate all but one variable and find that $C[\mathfrak{p}]_S$ sits in a short exact sequence

$$0 \longrightarrow C[\mathfrak{p}]_S \longrightarrow \mathbf{G}_{a,S} \xrightarrow{a-F^d} \mathbf{G}_{a,S} \longrightarrow 0$$

for some $a \in \mathcal{O}_{Y,\bar{s}}$, well-defined up to a unit. We claim that $a = \lambda^{q^d-1}$ (up to a unit). To see this, we compute the discriminant of the finite flat S-scheme $C[\mathfrak{p}]_S$ in two ways. On the one hand $C[\mathfrak{p}]_S$ is defined by the equation $X^{q^d} - aX$, with discriminant a^{q^d} (modulo squares of units). On the other hand, $C[\mathfrak{p}]$ is the $\mathfrak{p}$-torsion scheme of the Carlitz module and hence it is given by an equation

$$X^{q^d} + b_{d-1}X^{q^{d-1}} + \ldots + b_0 X$$

with $b_i \in A$, and with b_0 a generator of $\mathfrak{p}$. In this way we find that the discriminant equals $b_0^{q^d}$ (modulo squares of units). Comparing the two expressions we conclude that we can take $a = \lambda^{q^d-1}$, which proves the claim.

To finish the proof it suffices to observe that this short exact sequence is already defined over some étale neighbourhood $V \to Y$ of $\bar{s}$. □

Using this lemma we can now prove Proposition 2.

Proof of Proposition 2. Let V be as in the lemma and $U := Y - \mathfrak{P}$. Then $\{U, V\}$ is an étale cover of Y and it suffices to prove that the pull-backs of (6) to U_{et} and V_{et} are exact.

The pull-back to U_{et} is the sequence

$$0 \longrightarrow \mathrm{R}^1 f_* C[\mathfrak{p}]_U^{\mathrm{D}} \xrightarrow{\lambda} \mathrm{R}^1 f_* (A/\mathfrak{p})_U^{\mathrm{D}} \longrightarrow 0$$

which is exact because $\lambda\colon (A/\mathfrak{p})_U \to C[\mathfrak{p}]_U$ is an isomorphism of sheaves on U_{fl}.

For the exactness over V_{et}, consider the commutative square

$$\begin{array}{ccc}
\mathbf{G}_{a,V} & \xrightarrow{1-F^d} & \mathbf{G}_{a,V} \\
\downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda^{q^d}} \\
\mathbf{G}_{a,V} & \xrightarrow{\lambda^{q^d-1}-F^d} & \mathbf{G}_{a,V}
\end{array}$$

It extends to a map of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (A/\mathfrak{p})_V & \longrightarrow & \mathbf{G}_{a,V} & \xrightarrow{1-F^d} & \mathbf{G}_{a,V} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda^{q^d}} & & \\
0 & \longrightarrow & C[\mathfrak{p}]_V & \longrightarrow & \mathbf{G}_{a,V} & \xrightarrow{\lambda^{q^d-1}-F^d} & \mathbf{G}_{a,V} & \longrightarrow & 0
\end{array}$$

and without loss of generality we may assume that the leftmost vertical map is the one induced by λ. Now Theorem 5 (with k, F, and S replaced by $A/\mathfrak{p}$, F^d and V) yields a commutative diagram of sheaves of $A/\mathfrak{p}$-vector spaces on V_{et} with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{R}^1 f_* C[\mathfrak{p}]_V^{\mathrm{D}} & \longrightarrow & \Omega_V & \xrightarrow{\lambda^{q^d-1}-\mathrm{c}^d} & \Omega_V & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda^{q^d}} & & \downarrow{\scriptstyle \lambda} & & \\
0 & \longrightarrow & \mathrm{R}^1 f_*(A/\mathfrak{p})_V^{\mathrm{D}} & \longrightarrow & \Omega_V & \xrightarrow{1-\mathrm{c}^d} & \Omega_V & \longrightarrow & 0
\end{array}$$

(where by abuse of notation, we denote the canonical maps of sites $V_{\mathrm{fl}} \to V_{\mathrm{et}}$ and $Y_{\mathrm{fl}} \to Y_{\mathrm{et}}$ by the same symbol f.) This shows that on V_{et} we have an exact sequence

$$0 \longrightarrow \mathrm{R}^1 f_* C[\mathfrak{p}]_V^{\mathrm{D}} \xrightarrow{\lambda} \mathrm{R}^1 f_*(A/\mathfrak{p})_V^{\mathrm{D}} \longrightarrow \Omega_V/\lambda^{q^d}\Omega_V \xrightarrow{1-\mathrm{c}^d} \Omega_V/\lambda\Omega_V \longrightarrow 0,$$

so the pullback of (6) to V_{et} is exact. □

8. A candidate cohomology class

Let $\lambda \in R$ be a primitive $\mathfrak{p}$-torsion point of the Carlitz module. Consider the decomposition

$$1 \otimes \lambda = \sum_{n=1}^{q^d-1} \lambda_n$$

in $A/\mathfrak{p} \otimes_{\mathbf{Z}} \Gamma(Y - \mathfrak{P}, \mathcal{O}_Y^\times)$. In this section we will prove Theorem 3, which states that for $1 \leq n < q^d - 1$ we have

$$\lambda_n \in A/\mathfrak{p} \otimes \Gamma(Y, \mathcal{O}_Y^\times)$$

and that the following are equivalent

(1) $\mathfrak{p}$ divides BC_n;
(2) $\operatorname{dlog} \lambda_n$ lies in the kernel of $A/\mathfrak{p} \otimes_k \Omega_R \twoheadrightarrow \Omega_R/\mathfrak{P}^{q^d}\Omega_R$.

We start with the first assertion.

Proposition 3. *If $1 \leq n < q^d - 1$ then $\lambda_n \in A/\mathfrak{p} \otimes_{\mathbf{Z}} \Gamma(Y, \mathcal{O}^\times)$.*

Proof. For all integers n we have

$$\lambda_n = -\sum_{g \in G} \chi(g)^{-n} \otimes g\lambda.$$

If moreover n is not divisible by $q^d - 1$ then $\sum_{g\in G} \chi(g)^{-n} = 0$ so that we can rewrite the above identity as

$$\lambda_n = -\sum_{g \in G} \chi(g)^{-n} \otimes \frac{g\lambda}{\lambda}.$$

Since the point $\mathfrak{P}$ is fixed under G it follows that for all $g \in G$ one has that $g\lambda/\lambda$ has valuation 0 at $\mathfrak{P}$ and therefore for all $1 \leq n < q^d - 1$ we have

$$\lambda_n \in A/\mathfrak{p} \otimes \Gamma(Y, \mathcal{O}_Y^\times),$$

as was claimed. □

Now let $L_{\mathfrak{P}}$ be the completion of L at $\mathfrak{P}$ and $\mathfrak{m}$ the maximal ideal of its valuation ring $\mathcal{O}_{Y,\mathfrak{P}}^\wedge$. Note that $\mathfrak{m} = (\lambda)$.

Consider the quotient $\mathfrak{m}/\mathfrak{m}^{q^d}$. It carries two A-module structures:

(1) the *linear* action coming from the A-algebra structure of $\mathcal{O}^{\wedge}_{Y,\mathfrak{P}}$;
(2) the *Carlitz* action defined using φ.

Also, the Galois group G acts on $\mathfrak{m}/\mathfrak{m}^{q^d}$ and the action commutes with both A-module structures.

Lemma 2. *Both actions on $\mathfrak{m}/\mathfrak{m}^{q^d}$ factor over $A/\mathfrak{p}$.*

Proof. Note that $\mathfrak{p}\mathcal{O}^{\wedge}_{Y,\mathfrak{P}} = \mathfrak{m}^{q^d-1}$. In particular the assertion is immediate for the linear action. For the Carlitz action, consider a generator f of $\mathfrak{p}$. Then

$$\varphi(f) = a_0 + a_1 F + \cdots + a_{d-1}F^{d-1} + F^d$$

with $a_i \in \mathfrak{p}$ for all i. From this it follows that $\varphi(f)$ maps $\mathfrak{m} \subset \mathcal{O}^{\wedge}_{Y,\mathfrak{P}}$ into $\mathfrak{m}^{q^d}$, as desired. □

The Carlitz exponential series

$$e(z) = \sum_{n=1}^{\infty} e_n z^n \in K[[z]]$$

has the property that for all $n < q^d$ the coefficient e_n is $\mathfrak{p}$-integral, so the truncated and reduced exponential power series

$$\bar{e}(z) = \sum_{n=1}^{q^d-1} e_n z^n \in (A/\mathfrak{p})[[z]]/(z^{q^d})$$

defines a k-linear map

$$\bar{e}\colon \mathfrak{m}/\mathfrak{m}^{q^d} \to \mathfrak{m}/\mathfrak{m}^{q^d}$$

which is an isomorphism because it induces the identity map on the intermediate quotients $\mathfrak{m}^i/\mathfrak{m}^{i+1}$. Note that $\bar{e}$ is G-equivariant, as the coefficients e_i of the Carlitz exponential lie in K.

Lemma 3. *For all $x \in \mathfrak{m}/\mathfrak{m}^{q^d}$ and $a \in A$ we have $\bar{e}(ax) = \varphi(a)\bar{e}(x)$.*

Proof. In $K[[z]]$ we have the identity

$$e(tz) = te(z) + e(z)^q$$

of formal power series. Identifying coefficients on both sides we find that in $(A/\mathfrak{p})[[z]]/(z^{q^d})$ we have

$$\bar{e}(tz) = t\bar{e}(z) + \bar{e}(z)^q,$$

and we deduce that for all $a \in A$ and $x \in \mathfrak{m}/\mathfrak{m}^{q^d}$ we have $\bar{e}(ax) = \varphi(a)\bar{e}(x)$. □

Put $\bar{\pi} := \bar{e}^{-1}(\bar{\lambda})$, where $\bar{\lambda}$ is the image of $\lambda \in \mathfrak{m}$ in $\mathfrak{m}/\mathfrak{m}^{q^d}$.

Lemma 4. *For all $g \in G$ we have $g\bar{\pi} = \chi(g)\bar{\pi}$.*

In other words $\bar{\pi} \in \mathfrak{m}/\mathfrak{m}^{q^d}(\chi)$.

Proof of Lemma 4. Let $g \in G$ and $a \in A$ be so that a reduces to g in $G = (A/\mathfrak{p})^{\times}$. Since λ is a $\mathfrak{p}$-torsion point of the Carlitz module we have that

$$g\bar{\lambda} = \varphi(a)\bar{\lambda}.$$

Applying $\bar{e}^{-1}$ to both sides we find with Lemma 3 that

$$g\bar{\pi} = a\bar{\pi}$$

and by definition $a\bar{\pi}$ equals $\chi(g)\bar{\pi}$. □

Choose a lift $\pi \in \mathfrak{m}$ of $\bar{\pi}$ such that $g\pi = \chi(g)\pi$ for all g. Then π is a uniformizing element of $L_{\mathfrak{p}}$.

Proposition 4. *Let $1 \leq n < q^d - 1$. Then*

$$\mathrm{dlog}\, \lambda_n = (BC_n \pi^n + \delta)\, \mathrm{dlog}\, \pi$$

for some $\delta \in \mathfrak{m}^{n+q^d-1}$.

Proof. Since $\bar{\lambda} = \bar{e}(\bar{\pi})$ we have in $\mathcal{O}^{\wedge}_{Y,\mathfrak{P}}$ the identity

$$\lambda = \sum_{n=1}^{q^d-1} e_n \pi^n + \delta_1$$

for some $\delta_1 \in \mathfrak{m}^{q^d}$. Since $\mathrm{d}\pi^n = 0$ for any n divisible by q we find

$$\mathrm{d}\lambda = (1 + \delta_2)\mathrm{d}\pi$$

for some $\delta_2 \in \mathfrak{m}^{q^d}$. Dividing both expressions we find

$$\mathrm{dlog}\, \lambda = \left(\sum_{n=0}^{q^d-2} BC_n \pi^n + \delta_3 \right) \mathrm{dlog}\, \pi$$

for some $\delta_3 \in \mathfrak{m}^{q^d-1}$. Now the proposition follows from decomposing this identity in isotypical components, since $\mathrm{dlog}\, \pi$ is G-invariant and $g\pi = \chi(g)\pi$ for all $g \in G$. □

We can now finish the proof of Theorem 3.

Proof of Theorem 3. If $n > 1$ then the Theorem follows from the above proposition. If $n = 1$ we consider two cases. Either $q > 2$ and then $BC_1 = 0$ and $\mathrm{dlog}\, \lambda_1 = 0$, or else $q = 2$ and then $\mathfrak{p}$ does not divide BC_1 and from the above π-adic expansion we see that $\mathrm{dlog}\, \lambda_1$ does not map to zero in $\Omega_R / \mathfrak{P}^{q^d} \Omega_R$. In both cases the theorem holds. □

9. Vanishing of λ_n

Let W be the ring of Witt vectors of $A/\mathfrak{p}$. For $a \in (A/\mathfrak{p})^\times$ we denote by $\tilde{a} \in W^\times$ the Teichmüller lift of a. Also, we denote by $\tilde{\chi}\colon G \to W^\times$ the Teichmüller lift of the character $\chi\colon G \to (A/\mathfrak{p})^\times$. If M is a $W[G]$-module then it decomposes into isotypical components

$$M = \bigoplus_{n=1}^{q^d-1} M(\tilde{\chi}^n)$$

with G acting via $\tilde{\chi}^n$ on $M(\tilde{\chi}^n)$.

Put $U := W \otimes_{\mathbf{Z}} \Gamma(Y, \mathcal{O}_Y^\times)$ and let D be the W-module of degree zero W-divisors on $X - Y$. Then we have a natural inclusion $U \hookrightarrow D$ with finite quotient. Consider the decomposition of $1 \otimes \lambda \in W \otimes \Gamma(Y - \mathfrak{P}, \mathcal{O}_Y^\times)$ in isotypical components:

$$1 \otimes \lambda = \sum_{n=1}^{q^d-1} \tilde{\lambda}_n \quad \text{with} \quad \tilde{\lambda}_n \in W \otimes_{\mathbf{Z}} \Gamma(Y - \mathfrak{P}, \mathcal{O}_Y^\times)(\tilde{\chi}^n).$$

We have

$$\tilde{\lambda}_n = \sum_{g \in G} \chi(g)^{-n} \otimes g\lambda$$

and for $1 < n < q^d - 1$ we have that $\tilde{\lambda}_n$ lies in $U(\tilde{\chi}^n)$ and it maps to λ_n under the reduction map

$$U \longrightarrow\!\!\!\!\rightarrow A/\mathfrak{p} \otimes_{\mathbf{Z}} \Gamma(Y, \mathcal{O}_Y^\times).$$

If n is divisible by $q-1$ but not by $q^d - 1$, the W-modules $D(\tilde{\chi}^n)$ and $U(\tilde{\chi}^n)$ are free of rank one. In particular

$$\lambda_n = 0 \text{ if and only if } \frac{U(\tilde{\chi}^n)}{W\tilde{\lambda}_n} \neq 0,$$

and Theorem 4 follows from the following.

Proposition 5. *Let n be divisible by $q-1$ but not by q^d-1. Then the finite W-modules*

$$\frac{U(\tilde{\chi}^n)}{W\lambda_n}$$

and

$$W \otimes_{\mathbf{Z}} \operatorname{Pic} Y(\tilde{\chi}^n)$$

have the same length.

Proof. Let X be the canonical compactification of Y. Since we have a short exact sequence of W-modules

$$0 \longrightarrow \frac{D(\tilde{\chi}^n)}{U(\tilde{\chi}^n)} \longrightarrow W \otimes_{\mathbf{Z}} (\operatorname{Pic}^0 X)(\tilde{\chi}^n) \to W \otimes_{\mathbf{Z}} (\operatorname{Pic} Y)(\tilde{\chi}^n) \longrightarrow 0,$$

it suffices to show that

$$\frac{D(\tilde{\chi}^n)}{W\lambda_n} \text{ and } W \otimes_{\mathbf{Z}} (\operatorname{Pic}^0 X)(\tilde{\chi}^n)$$

have the same length. By Goss and Sinnott [8] the length of $W \otimes_{\mathbf{Z}} (\operatorname{Pic}^0 X)(\tilde{\chi}^n)$ is the p-adic valuation of $L(1, \tilde{\chi}^{-n}) \in W$. We will show that also the length of $D(\tilde{\chi}^n)/W\lambda_n$ equals the p-adic valuation of $L(1, \tilde{\chi}^{-n})$.

Since n is divisible by $q-1$, the representation $\tilde{\chi}^{-n}$ is unramified at ∞. Since all the points of X lying above ∞ are k-rational, the local L-factor at ∞ of $L(T, \tilde{\chi}^{-n})$ is $(1-T)^{-1}$. Since n is not divisible by $q^d - 1$, the representation is ramified at $\mathfrak{p}$ and hence the local L-factor at $\mathfrak{p}$ is 1. Recall that for a prime $\mathfrak{q} \subset A$ coprime with $\mathfrak{p}$ we have that $\chi(\mathrm{Frob}_{\mathfrak{q}})$ is the image of the monic generator of $\mathfrak{q}$ in $(A/\mathfrak{p})^\times$. Together with unique factorization in A we obtain

$$L(T, \tilde{\chi}^{-n}) = (1-T)^{-1} \sum_{a \in A_+, a \notin \mathfrak{p}} \tilde{a}^{-n} T^{\deg a},$$

where A_+ is the set of monic elements of A. In fact it is easy to see that for $m \geq d$ the coefficient of T^m in the sum vanishes, so we have

$$L(T, \chi^{-n}) = (1-T)^{-1} \sum_{a \in A_+^{<d}} \tilde{a}^{-n} T^{\deg a}, \tag{7}$$

where $A_+^{<d}$ is the set of monic elements of degree smaller then d.

Since n is divisible by $q-1$ we have

$$\sum_{a \in A_+^{<d}} \tilde{a}^{-n} T^{\deg a} = \frac{1}{q-1} \sum_{a \in A^{<d}} \tilde{a}^{-n} T^{\deg a}.$$

We conclude from (7) that

$$L(1,\tilde{\chi}^{-n}) = \frac{1}{q-1}\sum_{a\in A^{<d}} (\deg a)\tilde{a}^{-n}.$$

Consider the function

$$\deg\colon G \to \{0,1,\ldots,d-1\}$$

which maps $g \in G$ to the degree of its unique representative in $A^{<d}$. Then the above identity can be rewritten as

$$L(1,\tilde{\chi}^{-n}) = \frac{1}{q-1}\sum_{g\in G} (\deg g)\tilde{g}^{-n}.$$

By [4, *p.* 372] there is a point in $X - Y$ with associated valuation v and integers u, w with $(u,p) = 1$ such that

$$v(g\lambda) = u \deg g + w$$

for all $g \in G$. The valuation v extends to an isomorphism of W-modules

$$v\colon D(\tilde{\chi}^n) \to W,$$

and we have

$$\begin{aligned} v(\lambda_n) &= \sum_{g\in G} \tilde{g}^{-n} v(g\lambda) \\ &= u(q-1)L(1,\tilde{\chi}^{-n}) + w\sum_{g\in G}\tilde{g}^{-n} \\ &= u(q-1)L(1,\tilde{\chi}^{-n}). \end{aligned}$$

In particular, the length of $D(\tilde{\chi}^n)/\lambda_n$ is the p-adic valuation of $L(1,\tilde{\chi}^{-n})$ and the proposition follows. $\square$

10. Complement: the class module of Y

Let L be an arbitrary finite extension of K and R the integral closure of A in L. Put $Y = \operatorname{Spec} R$. In [16] and [15] we have given several equivalent definitions of a finite A-module $H(C/Y)$ depending on Y, that is analogous to the class group of a number field. One of these definitions is the following.

Let X be the canonical compactification of Y and let ∞ be the divisor on X of zeroes of $1/t \in L$. (This is also the inverse image of the divisor ∞ on $\mathbf{P}^1$.) Then $H(C/Y)$ is defined by the exact sequence

$$A \otimes_k \mathrm{H}^1(X, \mathcal{O}_X) \xrightarrow{\partial} A \otimes_k \mathrm{H}^1(X, \mathcal{O}_X(\infty)) \longrightarrow H(C/Y) \longrightarrow 0, \tag{8}$$

where

$$\partial = 1 \otimes (t + \mathrm{F}) - t \otimes 1.$$

Theorem 6. *Let $I \subset A$ be a nonzero ideal. Then there is a natural isomorphism*

$$\mathrm{H}^1(Y_{\mathrm{fl}}, C[I]^{\mathrm{D}})^\vee \xrightarrow{\sim} H(C/Y) \otimes_A A/I$$

where $(-)^\vee$ denotes the k-linear dual.

Proof. The starting point of the proof is the exact sequence of sheaves of A-modules

$$0 \longrightarrow A \otimes_k \mathbf{G}_a \xrightarrow{\partial} A \otimes_k \mathbf{G}_a \xrightarrow{\alpha} C \longrightarrow 0$$

with $\partial(a \otimes f) = a \otimes (f^q + tf) - ta \otimes f$ and with $\alpha(a \otimes f) = \varphi(a)f$. From this we derive a short exact sequence

$$0 \longrightarrow C[I]_Y \longrightarrow A/I \otimes_k \mathbf{G}_a \xrightarrow{\partial} A/I \otimes_k \mathbf{G}_a \longrightarrow 0.$$

Using Theorem 5 we obtain a dual resolution:

$$0 \longrightarrow \mathrm{R}^1 f_* C[I]^{\mathrm{D}} \longrightarrow A/I \otimes_k \Omega_Y \xrightarrow{\partial^*} A/I \otimes_k \Omega_Y \longrightarrow 0$$

of sheaves of A-modules on Y_{et}, where $\partial^* = 1 \otimes (t + \mathrm{c}) - t \otimes 1$. Since $\mathrm{R}^i f_* C[I]^{\mathrm{D}} = 0$ for $i \neq 1$, taking global sections we obtain an exact sequence of A-modules

$$0 \longrightarrow \mathrm{H}^1(Y_{\mathrm{fl}}, C[I]^{\mathrm{D}}) \longrightarrow A/I \otimes_k \Gamma(Y, \Omega_Y) \xrightarrow{\partial^*} A/I \otimes_k \Gamma(Y, \Omega_Y). \tag{9}$$

Now we claim that the natural inclusion of the complex

$$A/I \otimes_k \Gamma(X, \Omega_X(-\infty)) \xrightarrow{\partial^*} A/I \otimes_k \Gamma(X, \Omega_X)$$

in the complex

$$A/I \otimes_k \Gamma(Y, \Omega_Y) \xrightarrow{\partial^*} A/I \otimes_k \Gamma(Y, \Omega_Y)$$

is a quasi-isomorphism. Indeed, the quotient has a filtration with intermediate quotients of the form

$$A/I \otimes_k \frac{\Gamma(X, \Omega_X(n\infty))}{\Gamma(X, \Omega_X((n-1)\infty))} \xrightarrow{\partial^*} A/I \otimes_k \frac{\Gamma(X, \Omega_X((n+1)\infty))}{\Gamma(X, \Omega_X(n\infty))}$$

with $n \in \mathbf{Z}_{\geq 0}$. On these intermediate quotients we have that $1 \otimes \mathrm{c}$ and $t \otimes 1$ are zero, so that $\partial^* = 1 \otimes t$, which is an isomorphism.

Hence we obtain from (9) a new exact sequence

$$0 \longrightarrow \mathrm{H}^1(Y_{\mathrm{fl}}, C[I]^{\mathrm{D}}) \longrightarrow A/I \otimes_k \Gamma(X, \Omega_X(-\infty)) \xrightarrow{\partial^*} A/I \otimes_k \Gamma(X, \Omega_X).$$

Under Serre duality the q-Cartier operator c on Ω_X is adjoint to the q-Frobenius F on $\mathcal{O}_X$, so we obtain a dual exact sequence

$$A/I \otimes_k \mathrm{H}^1(X, \mathcal{O}_X) \xrightarrow{\partial} A/I \otimes_k \mathrm{H}^1(X, \mathcal{O}_X(\infty)) \longrightarrow \mathrm{H}^1(Y_{\mathrm{fl}}, C[I]^{\mathrm{D}})^\vee \longrightarrow 0.$$

Theorem 6 now follows by comparing this sequence with the sequence obtained by reducing (8) modulo I. □

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Mathematisch Instituut, P.O. Box 9512, 2300 RA Leiden, The Netherlands
E-mail address: `lenny@math.leidenuniv.nl`